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The canonical form of the $E \otimes \epsilon$ Jahn–Teller Hamiltonian

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Abstract. The $E \otimes \epsilon$ Jahn–Teller Hamiltonian in its Bargmann–Fock representation is transformed by the Birkhoff method into a canonical form in which all regular singularities between zero and infinity have been removed. The resulting equation is of the Kummer type and identical to the previously obtained canonical form of the Rabi Hamiltonian (Szopa M and Ceulemans A 1996 *J. Math. Phys.* **37** 5402). The isolated exact solutions of the $E \otimes \beta$, $E \otimes \epsilon$ and $\Gamma_8 \otimes \tau_2$ Jahn–Teller Hamiltonians are identified as special symmetries of the canonical form.

1. Introduction

The $E \otimes \epsilon$ Jahn–Teller (JT) Hamiltonian describes the vibronic coupling between a twofold degenerate electronic level and a pair of degenerate vibrational modes. In 1958 Longuet-Higgins *et al* [2] carried out the first diagonalization of this Hamiltonian within the adiabatic approximation. In the present day, with modern computational means, numerical solutions of this problem may routinely be obtained with any desired degree of accuracy [3].

For some values of the coupling parameter the eigenvalues of the system appear to be rational numbers, corresponding to intersections with so-called baselines. This was first noted in numerical work by Moffit and Thorson [4] on the related $\Gamma_8 \otimes \tau_2$ system. In a remarkable tour de force Judd [5] succeeded in establishing finite-order equations for determining the strength of the coupling for which the eigenvalue lies on a baseline. Further progress in the analytical treatment was mainly due to Reik *et al* [6, 7], who reformulated the problem in Bargmann–Fock space and conjectured the existence of general exact solutions.

In the present paper we continue to explore the problem in the Bargmann–Fock space, using a different approach, however. While Reik was looking for natural expansion functions which incorporate the singularities of the problem, we choose to apply the Birkhoff transformation method [1, 8] in which all finite singularities are reduced to only one singularity at the origin. This results in a canonical form of the $E \otimes \epsilon$ JT Hamiltonian which appears to be identical to the previously derived canonical form of the Rabi (or equivalently $E \otimes \beta$ Jahn–Teller) Hamiltonian [1]. The Juddian isolated exact solutions of the JT problem and the Kuś isolated exact solutions of the Rabi problem [9] are found to correspond to resonant cases of the canonical form.

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2. The $E\otimes\epsilon$ Jahn–Teller Hamiltonian

The Hamiltonian of the $E\otimes\epsilon$ Jahn–Teller system describes a two-level fermionic subsystem coupled to two boson modes:

$$H=\sum_{i=1}^{2}a_i^+a_i+\mu\sigma_3+\sqrt{2}\lambda[\sigma^+(a_1+a_2^+)+\sigma^-(a_1^++a_2)] \tag{1}$$

where a_i^+ and a_i are boson field creation and annihilation operators

$$\left[a_i^+,a_j^+\right]=\left[a_i,a_j\right]=0 \qquad \left[a_i,a_j^+\right]=\delta_{i,j} \tag{2}$$

$\sigma^{\pm}=\frac{1}{2}(\sigma_1\pm\mathrm{i}\sigma_2)$ and $\sigma_1,\sigma_2,\sigma_3$ are Pauli matrices. The parameter 2μ is the level separation of the fermionic states and $\sqrt{2}\lambda$ is the fermion–boson field coupling constant. We assume that it is non-trivial $\lambda\neq 0$. The boson vacuum state in (1) is shifted to zero energy.

Closely following the analysis of Reik [7] the Hamiltonian (1) is divided into two parts

$$H=J+H_1 \tag{3}$$

where

$$J=a_1^+a_1-a_2^+a_2+\tfrac{1}{2}\sigma_3 \tag{4}$$

represents the angular momentum of the system and the remaining part is

$$H_1=2a_2^+a_2+(\mu-\tfrac{1}{2})\sigma_3+\sqrt{2}\lambda[\sigma^+(a_1+a_2^+)+\sigma^-(a_1^++a_2)]. \tag{5}$$

Both parts commute with the full Hamiltonian. The eigenproblem of the angular momentum part can be easily solved and reads

$$J|\psi\rangle_{j+\frac{1}{2}}=(j+\tfrac{1}{2})|\psi\rangle_{j+\frac{1}{2}} \qquad j=0,1,2,\ldots \tag{6}$$

where

$$|\psi\rangle_{j+\frac{1}{2}}=(a_1^+)^j\phi_1(a_1^+a_2^+)|0\rangle|\uparrow\rangle+(a_1^+)^{j+1}\phi_2(a_1^+a_2^+)|0\rangle|\downarrow\rangle \tag{7}$$

is the eigenfunction. Here $|0\rangle$ is the vacuum state for both bosons, $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenstates of the σ_3 operator and ϕ_1 and ϕ_2 are arbitrary functions (at least analytic). The ansatz in (7) was proposed by Reik and does not treat the two boson modes on an equal footing. Later on we will replace it by a more symmetrical form which greatly facilitates further treatment. Because the operators H and J commute, the functions (7) are also eigenfunctions of the Hamiltonian (1):

$$H|\psi\rangle_{j+\frac{1}{2}}=E|\psi\rangle_{j+\frac{1}{2}} \tag{8}$$

and of its component H_1:

$$H_1|\psi\rangle_{j+\frac{1}{2}}=(E-j-\tfrac{1}{2})|\psi\rangle_{j+\frac{1}{2}}. \tag{9}$$

The wavefunctions of the $E\otimes\epsilon$ JT system are thus seen to be characterized by half-integral rotational quantum numbers, which is a direct consequence of the geometrical Berry phase [8–10]. Consequently, the solution of the stationary Schrödinger equation of the Hamiltonian can be reduced to an eigenequation (9) of H_1.

In what follows we use the Bargmann–Fock representation of the problem. To do this, the boson field creation operators are replaced by multiplication operators and the annihilation operators by differentiation ones:

$$a_i^+\longrightarrow\xi_i \qquad a_i\longrightarrow\frac{\partial}{\partial\xi_i} \tag{10}$$

$i = 1, 2$, with respect to complex variables ξ_1 and ξ_2. The trial eigenfunction (7) is of the form

$$|\psi\rangle_{j+\frac{1}{2}} = (\xi_1)^j \phi_1(\xi)|\uparrow\rangle + (\xi_1)^{j+1}\phi_2(\xi)|\downarrow\rangle \tag{11}$$

where $\xi = \xi_1 \cdot \xi_2$. The insertion of (11) into (9) yields a system of two ordinary, linear, first-order differential equations for the functions $\phi_1(\xi)$ and $\phi_2(\xi)$:

$$\begin{aligned}
&(\mu - E + j)\phi_1(\xi) + 2\xi \frac{\mathrm{d}\phi_1(\xi)}{d\xi} + \sqrt{2}\lambda \left[(\xi + j + 1)\phi_2(\xi) + \xi \frac{\mathrm{d}\phi_2(\xi)}{\mathrm{d}\xi} \right] = 0 \\
&\sqrt{2}\lambda \left[\phi_1(\xi) + \frac{\mathrm{d}\phi_1(\xi)}{\mathrm{d}\xi} \right] - (\mu + E - j - 1)\phi_2(\xi) + 2\xi \frac{\mathrm{d}\phi_2(\xi)}{\mathrm{d}\xi} = 0.
\end{aligned} \tag{12}$$

Note that in this form, already obtained by Reik [7], the two variables ξ_1 and ξ_2 exist only in their product form $\xi = \xi_1 \cdot \xi_2$.

Solutions of this system describe a quantum mechanical state of H provided that $\phi_1(\xi)$ and $\phi_1(\xi)$ belong to the Bargmann–Fock space, i.e. they are complete and normalizable:

$$\int \overline{\phi_i(\xi)}\phi_i(\xi) \exp(-\bar{\xi}\xi)\ \mathrm{d}(\mathrm{Re}\,\xi)\ \mathrm{d}(\mathrm{Im}\,\xi) < \infty \qquad i = 1, 2. \tag{13}$$

Reik [7] attempted to solve (12) by a suitable choice of natural expansion functions which incorporate the singular points of these equations. Here we take a different direction by introducing a transformation which removes the singular points.

Our first step is to symmetrize the system (12). By means of the substitutions $\xi = z^2$ and

$$\begin{aligned}
\phi_1(\xi) &= \frac{\psi_1(z) + \psi_2(z)}{2} z^{-j} \\
\phi_2(\xi) &= \frac{\psi_1(z) - \psi_2(z)}{2} z^{-j-1}
\end{aligned} \tag{14}$$

we obtain the symmetrical form of (12):

$$\begin{aligned}
\frac{\mathrm{d}\psi_1(z)}{\mathrm{d}z} &= \left(\frac{-\frac{1}{2}}{z} + \frac{E + \lambda^2 + \frac{1}{2}}{z + \lambda} - \lambda \right) \psi_1(z) + \left(\frac{j + \frac{1}{2}}{z} - \frac{\mu + j + \frac{1}{2}}{z + \lambda} \right) \psi_2(z) \\
\frac{\mathrm{d}\psi_2(z)}{\mathrm{d}z} &= \left(\frac{j + \frac{1}{2}}{z} - \frac{\mu + j + \frac{1}{2}}{z - \lambda} \right) \psi_1(z) + \left(\frac{-\frac{1}{2}}{z} + \frac{E + \lambda^2 + \frac{1}{2}}{z - \lambda} + \lambda \right) \psi_2(z).
\end{aligned} \tag{15}$$

The solutions $\psi_1(z)$ and $\psi_2(z)$ represent physical states if the corresponding $\phi_1(\xi)$ and $\phi_2(\xi)$ are complete and obey the normalizability condition. The latter means that $\psi_i \circ \sqrt{\cdot}$ (the composition of ψ_i with the square root) obey (13).

The symmetry of (15) implies that for a given solution $\binom{\psi_1(z)}{\psi_2(z)}$ the pair $\binom{\psi_2(-z)}{\psi_1(-z)}$ is also a solution. In combination with the definition in (14), it follows that the eigenfunctions adopt a fixed parity with regard to this symmetry which only depends on the angular momentum quantum number j:

$$\begin{pmatrix} \psi_1(-z) \\ \psi_2(-z) \end{pmatrix} = (-1)^j \begin{pmatrix} \psi_2(z) \\ \psi_1(z) \end{pmatrix}. \tag{16}$$

Note that for the unphysical value of $j = -\frac{1}{2}$, which corresponds to a vanishing angular momentum eigenvalue (6), and for $\mu = 0$ the system (15) becomes uncoupled and can be solved exactly. The solutions of such a simplified case yield the quantization condition for E:

$$E + \lambda^2 + \tfrac{1}{2} = \nu \tag{17}$$

where ν is a non-negative integer. These quantized values of $E + \lambda^2$ correspond to the celebrated Judd baselines [5].

In the physical case however, we deal with the situation that the coupling term λ of the Hamiltonian (1) produces regular singularities, which, apart from $z = 0$, occur at $z = \lambda$ and $z = -\lambda$. Moreover, the system (15) has an irregular singularity of rank $q + 1 = 1$ at infinity [13].

3. The transformation to a canonical form

To simplify the singularity structure we now transform the system (15) into a canonical form with only one finite singularity at the origin by applying the Birkhoff theorem as explained in our previous paper [1]. Expanding coefficients of the system (15) in a Laurent series and applying equation (14) of [1] we find the canonical form of the eigenproblem under investigation:

$$\begin{aligned} z\frac{\mathrm{d}}{\mathrm{d}z}\Psi_1(z) &= \left(E - \lambda z + \lambda^2\right)\Psi_1(z) + \left(-\mu - 2\lambda a_{12}^{(1)}\right)\Psi_2(z) \\ z\frac{\mathrm{d}}{\mathrm{d}z}\Psi_2(z) &= \left(-\mu + 2\lambda a_{21}^{(1)}\right)\Psi_1(z) + \left(E + \lambda z + \lambda^2\right)\Psi_2(z) \end{aligned} \qquad (18)$$

where $a_{r,s}^{(1)}$ are first-order expansion coefficients of the matrix of complex functions $a_{rs}(z) = \sum_{k=0}^{\infty} a_{rs}^{(k)}/z^k$, $a_{rs}^{(0)} = \delta_{rs}$, which are defined by the linear transformation

$$\psi_r(z) = \sum_{s=1}^{2} a_{rs}(z)\Psi_s(z) \qquad r = 1, 2 \qquad (19)$$

leading from the symmetric form (15) to the canonical form (18).

In what follows we assume that the solutions of the canonical system (18) have the same parity properties as the solutions of the symmetrical system (15). It implies that the transformation matrix functions must be symmetric in the following sense:

$$a_{ij}(z) = a_{[i+1][j+1]}(-z) \qquad i, j = 1, 2 \qquad (20)$$

where $[t]$ is by definition 1 for odd t and 2 for even t.

Note that in our analysis we do not postulate any form of the transformation matrix; in fact we do not know it. What we know (by the Birkhoff theorem [8]) is that there exists a matrix $\{a_{ij}(z)\}$ such that the substitution of (19) in (15) leads to the transformed system (18). This system depends on the Hamiltonian parameters μ, λ and two constants $a_{12}^{(1)}$ and $a_{21}^{(1)}$. Due to symmetry, equation (20), only one of them, e.g. $a_{12}^{(1)}$, is independent. As we show later, apart from the angular momentum j, the solutions of the system can be labelled by Judd's baseline quantum number ν, equation (17).

The constant $a_{12}^{(1)}$ also defines the transformation matrix via the system of recurrence equations

$$\sum_{i=0}^{l}\left(\left\{a_{rk}^{(l-i)}\right\}\left\{P_{ks}^{(1-i)}\right\} - \left\{p_{rk}^{(-i)}\right\}\left\{a_{ks}^{(l-i)}\right\}\right) = (l-1)\left\{a_{rs}^{(l-1)}\right\} \qquad (21)$$

where $l = 2, 3\ldots$, $p_{rs}^{(k)}$ are defined by the expansion $p_{rs}(z) = \sum_{k=-\infty}^{0} p_{rs}^{(k)} z^k$ of the coefficients of the symmetric system (15) and $P_{rs}(k)$ are the polynomial coefficients $P_{rs}(z) = \sum_{k=0}^{1} P_{rs}^{(k)} z^k$ of the canonical system (18) (the $\{a_{rs}^{(i)}\}$ and $\{P_{rs}^{(i)}\}$ with negative i are zero by definition).

The important formula establishing the relation between the energy and the $a_{12}^{(1)}$ parameter is the indicial equation

$$\rho_{1,2} = E + \lambda^2 \pm A \tag{22}$$

where $A = \mu + 2\lambda a_{12}^{(1)}$ and the index 1 (2) corresponds to $+$ $(-)$. The necessary condition for the solutions of (18) to be physical (i.e. corresponding to ϕ_i, $i = 1, 2$ in the Bargmann–Fock space) is that at least one of the roots ρ_i of (22) must be a non-negative integer. As we show later, the integer ρ is determined by the quantum numbers of a particular solution. In a way therefore, determination of $a_{12}^{(1)}$ as a function of the system parameters μ, λ and the quantum numbers is equivalent to the determination of the energy spectrum.

An interesting property of the canonical form of the $E \otimes \epsilon$ Jahn–Teller eigenproblem is that the system (18) is identical to that obtained by us for the Rabi Hamiltonian [1]. Consequently, the formal solutions of the two systems are represented by the same functions. The system (18) can be reduced to a confluent hypergeometric (or Kummer) equation of second order. The general solutions for $\Psi_1(z)$ and $\Psi_2(z)$ are combinations of two functions

$$\begin{aligned}\Psi_1(z) &= \exp(\lambda z)[C_1\, z^{\rho_1}\, {}_1F_1(1+A, 1+2A; -2\lambda z) + C_2\, z^{\rho_2}\, {}_1F_1(1-A, 1-2A; -2\lambda z)] \\ \Psi_2(z) &= \exp(-\lambda z)[-C_1\, z^{\rho_1}\, {}_1F_1(1+A, 1+2A; 2\lambda z) + C_2\, z^{\rho_2}\, {}_1F_1(1-A, 1-2A; 2\lambda z)]\end{aligned} \tag{23}$$

where C_1 and C_2 are arbitrary constants and ${}_1F_1(a, c; z)$ is the confluent series (Kummer function) [14].

Note that the Kummer functions are complete and the indicial equation is really the only criterion for the completeness of the solutions (23). The functions $\Psi_{1,2} \circ \sqrt{\cdot}$ are normalized in the sense of (13), which means that the second necessary condition for them to represent physical states is fulfilled [1]. The sufficient condition for (23) to represent a physical state is that the corresponding $\phi_i(\xi)$, $i = 1, 2$, obtained by (19) and (14) belong to the Bargmann–Fock space. This can be achieved by the proper choice of the parameter $a_{12}^{(1)}$. Specific cases where this can be done are discussed in the next section.

4. The isolated exact solutions

In this section we examine some specific solutions for the transformation matrix leading to the exact solutions of the initial $E \otimes \epsilon$ Jahn–Teller problem.

The choice of a proper $a_{12}^{(1)}$ parameter for a given μ and λ is not possible in general. This is because the transformation matrix is determined by $a_{12}^{(1)}$ only via a set of recurrence relations (21), which in general do not lead to a compact form of this matrix allowing one to determine whether the corresponding solutions of the initial system ϕ_i, $i = 1, 2$ belong to the Bargmann–Fock space.

In some cases, however, the system of recurrence equations (21) can be solved exactly. These correspond to the assumption, that the expansion of the transformation matrix coefficients in negative powers of z terminates. If, for example, we assume that the only non-vanishing coefficients are $a_{ij}^{(1)}$ (i.e. $a_{ij}^{(2)} = a_{ij}^{(3)} = \cdots = 0$) then the system (21) can be reduced to only four equations:

$$\begin{aligned}&a_{11}^{(1)} + (2\lambda a_{12}^{(1)} + 2\mu)a_{12}^{(1)} = \nu\lambda \\ &(2\lambda a_{12}^{(1)} + 2\mu)a_{11}^{(1)} + a_{12}^{(1)} = -\lambda(j + \mu + \tfrac{1}{2}) \\ &(j + \mu + \tfrac{1}{2})a_{11}^{(1)} + \nu a_{12}^{(1)} = -\lambda(j + \mu + \tfrac{1}{2}) \\ &\nu a_{11}^{(1)} + (j + \mu + \tfrac{1}{2})a_{12}^{(1)} = \nu\lambda.\end{aligned} \tag{24}$$

The solution of the system (24) shows that all its physical solutions (i.e. corresponding to $j = 0, 1, 2 \ldots$) lie on the first Juddian line $\nu = 1$. The coupling parameter A is found to be equal to the total angular momentum, $A = j + \frac{1}{2}$, and the transformation matrix is

$$\left\{a_{rs}(z)\right\} = \begin{pmatrix} 1 + \frac{\lambda}{z} \frac{1 + (j + \mu + \frac{1}{2})^2}{1 - (j + \mu + \frac{1}{2})^2} & -\frac{2\lambda}{z} \frac{j + \mu + \frac{1}{2}}{1 - (j + \mu + \frac{1}{2})^2} \\ \frac{2\lambda}{z} \frac{j + \mu + \frac{1}{2}}{1 - (j + \mu + \frac{1}{2})^2} & 1 - \frac{\lambda}{z} \frac{1 + (j + \mu + \frac{1}{2})^2}{1 - (j + \mu + \frac{1}{2})^2} \end{pmatrix}. \qquad (25)$$

Simultaneously with the expression (25) the system (24) yields one more additional condition for λ^2, namely

$$\lambda^2 = \frac{(j - \mu + \frac{1}{2})(j + \mu + \frac{3}{2})(j + \mu - \frac{1}{2})}{4(j + \mu + \frac{1}{2})} \qquad (26)$$

which coincides with the intersection formula obtained by Reik [6]. The above constraint means that the solutions, which can be obtained by (25), are discrete points lying on the first Juddian line $\nu = 1$, corresponding to arbitrary angular momentum $j = 0, 1, 2 \ldots$, such that there is a balance between λ and μ described by (26). Now we check whether these solutions represent physical states. The necessary condition, i.e. the indicial equation, yields $\rho_1 = 1 + j$ (because $A = j + \frac{1}{2}$), which is always a positive integer. The other root is then found to be $\rho_2 = -j$, which, as a negative integer ($j = 1, 2, \ldots$), does not give rise to a physically acceptable solution; for $j = 0$ this solution is not acceptable either, since the Kummer function ${}_1F_1(\frac{1}{2}, 0; -2\lambda z)$ is not defined.

The solution (23), for half-integral A, is therefore one dimensional with $C_2 = 0$. The explicit formula for the transformation matrix (25) and substitution (14) allow one to find the solutions ϕ_1 and ϕ_2 of the initial problem (12) which contains only even powers of z. It can further be checked that $\phi_1(\xi)$ and $\phi_2(\xi)$ are complete and normalizable. This is the sufficient condition for (23) to represent physical states.

In a similar way one can find solutions of the system (21) under the assumption that the only non-vanishing expansion elements are of first and second order (i.e. $a_{ij}^{(3)} = a_{ij}^{(4)} = \cdots = 0$). The system then reduces to a set of six equations which have solutions along the second Juddian baseline $\nu = 2$. In this case $A = -(j + \frac{1}{2})$ and the second root of the indicial equation is always a positive integer $\rho_2 = 2 + j$. The physical solutions are again one dimensional and correspond to (23) with $C_1 = 0$. The additional condition for λ and μ to represent isolated exact solutions for arbitrary $j = 1, 2, \ldots$ is

$$32\lambda^4 - \left[\frac{(2L - 4M)(4 - L^2)}{L} + 2(4 + L^2 + 4LM)\right]\lambda^2 + 4 - L^2 - (4 - L^2)M^2 = 0 \qquad (27)$$

where $L = j + \mu + \frac{1}{2}$ and $M = j - \mu + \frac{1}{2}$. This again coincides with the criterion for the second intersection line [6].

In this way one can show in general that if the transformation matrix $\{a_{ij}(z)\}$ has a finite expansion of order ν, $\nu = 1, 2, \ldots$, then the system of recurrence equations (21) is finite of order $2(\nu + 1)$ and the corresponding eigenstates of the Hamiltonian (1) lie on the νth Juddian baseline. The transformed eigenfunction is one dimensional and is given by (23), where $C_{[\nu+1]} = 0$, $A = (-1)^{\nu+1}(j + \frac{1}{2})$ and $\rho_{[\nu]} = \nu + j$. Its parity alternates with j in the same way as the parity of the original functions in (16). The additional condition for λ and ν is in general a polynomial of order 2ν in λ. All these solutions are in agreement with numerical values found for $\mu = 0$ [5].

5. Discussion

In this paper it has been shown that both the Rabi and Jahn–Teller systems evolve from a common canonical form, which is found to be an exactly solvable Kummer equation. The transformation to the canonical form introduces an additional free parameter, $a_{12}^{(1)}$ or A, the value of which must be determined by the physical requirements of completeness and normalizability. A is further linked to the energy by the indicial equation (22). In general A cannot be evaluated in a closed form, except for isolated cases where the transforming series expansion terminates. In these cases A is found to be equal to the total angular momentum of the system

$$A = \pm(j + \tfrac{1}{2}) \tag{28}$$

where j is the bosonic part of the angular momentum and $\frac{1}{2}$ refers to the fermionic part.

As shown in our previous paper [1], the $E \otimes \beta$ Rabi Hamiltonian with the exact Kuś solutions [9] corresponds to the rotation-free case with $j = -\frac{1}{2}$, and hence $A = 0$. In this high-symmetry case the exact solutions form degenerate pairs of opposite parity lying on integral baselines.

The $E \otimes \epsilon$ Jahn–Teller Hamiltonian with the Juddian exact solutions [5] is characterized by $j = 0, 1, 2, \ldots$ and hence half-integral A values. In this case the indicial equation can have two integral roots but only one of the associated Kummer functions will describe a physical state. Therefore there will be no extra degeneracy except for that induced by time reversal.

For completeness we should also mention the $\Gamma_8 \otimes \tau_2$ Jahn–Teller Hamiltonian [5, 4], which leads to an entirely analogous treatment, but with $j = \frac{1}{2}, \frac{3}{2}, \ldots$. Again solutions are expected to be one dimensional under the Bargmann–Fock constraints. They will lie on integral baselines as in the Rabi case.

The analysis thus shows that the exact solutions naturally arise as special solutions of the canonical form, covering all possible cases where the parameter A adopts half-integral or integral values. To show the particular nature of these solutions it is instructive to rewrite the Kummer equation

$$\left[z^2 \frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left(1 - 2\left(E + \lambda^2\right)\right) z \frac{\mathrm{d}}{\mathrm{d}z} + \left(\left(E + \lambda^2\right)^2 - A^2 + \lambda z - \lambda^2 z^2\right)\right] \Psi_1(z) = 0 \tag{29}$$

in a 'gauge' equivalent form as

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{A^2 - \frac{1}{4}}{2z^2} - \frac{\lambda}{2z} + \frac{\lambda^2}{2}\right] \hat{\Psi}_1(z) = 0. \tag{30}$$

The 'gauge' transformation is defined by

$$\Psi_1(z) = \hat{\Psi}_1(z)\, \mathrm{e}^{\int \mathcal{A}(z)\,\mathrm{d}z} \tag{31}$$

where the 'gauge' potential is $\mathcal{A}(z) = (2(E + \lambda^2) - 1)/2z$. The transformation (31) is not unitary and changes the asymptotic behaviour of the wavefunction; therefore, following Shifman [15], we use quotation marks to distinguish it from the standard gauge transformation.

The Kummer equation in the form (30), at least for z real, can be recognized as a Schrödinger equation for a Coulomb potential with a centrifugal barrier. For $A = \pm(j + \frac{1}{2})$ the angular momentum constant of this barrier, which appears as $A^2 - \frac{1}{2}$ in (30), can at once be identified as the angular momentum associated with the bosonic part, namely

$$A^2 - \tfrac{1}{4} = j(j + 1). \tag{32}$$

In this way the isolated exact solutions of the JT Hamiltonians are put in correspondence with the special symmetries of the canonical form.

In conclusion, we have shown that there exists a matrix transformation that maps the JT Hamiltonian into an exactly solvable Kummer problem. Whenever the transformation matrix is finite this mapping offers a simple direct method for obtaining exact solutions of the initial Hamiltonian. The isolated solutions found in this way are seen to correspond to special quantizations of the canonical form.

These results are reminiscent of the properties of quasi-exactly solvable systems, that have recently received much attention [16]. It seems this would imply that the Reik conjecture is false. However, before such a claim could be validated, one would have to await a more general theory of quasi-exact solvability in systems of dimension higher than one.

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